Appendix

This Appendix contains additional results for the *Reassessing Growth Vulnerability* paper.

1 Additional Replication Results

Adrian et al. (2019) investigate the conditional quantile of the one-quarter ahead and one-year ahead GDP growth distributions as a function of current GDP growth and the NFCI using quarterly data. Specifically, the dependent variable would be either one-quarter ahead annualized GDP growth, denoted as y_{t+1}, or annualized average GDP growth between $t+1$ and $t+4$, denoted as y_{t+4}. Then, the τ conditional quantile of y_{t+h}, $h = \{1, 4\}$, is

$$Q_{y_{t+h}|x_t}(\tau) = x_t' \beta_\tau(h).$$

Adrian et al. (2019) consider the QR model of Koenker & Bassett (1978) which minimizes the following quantile loss function:

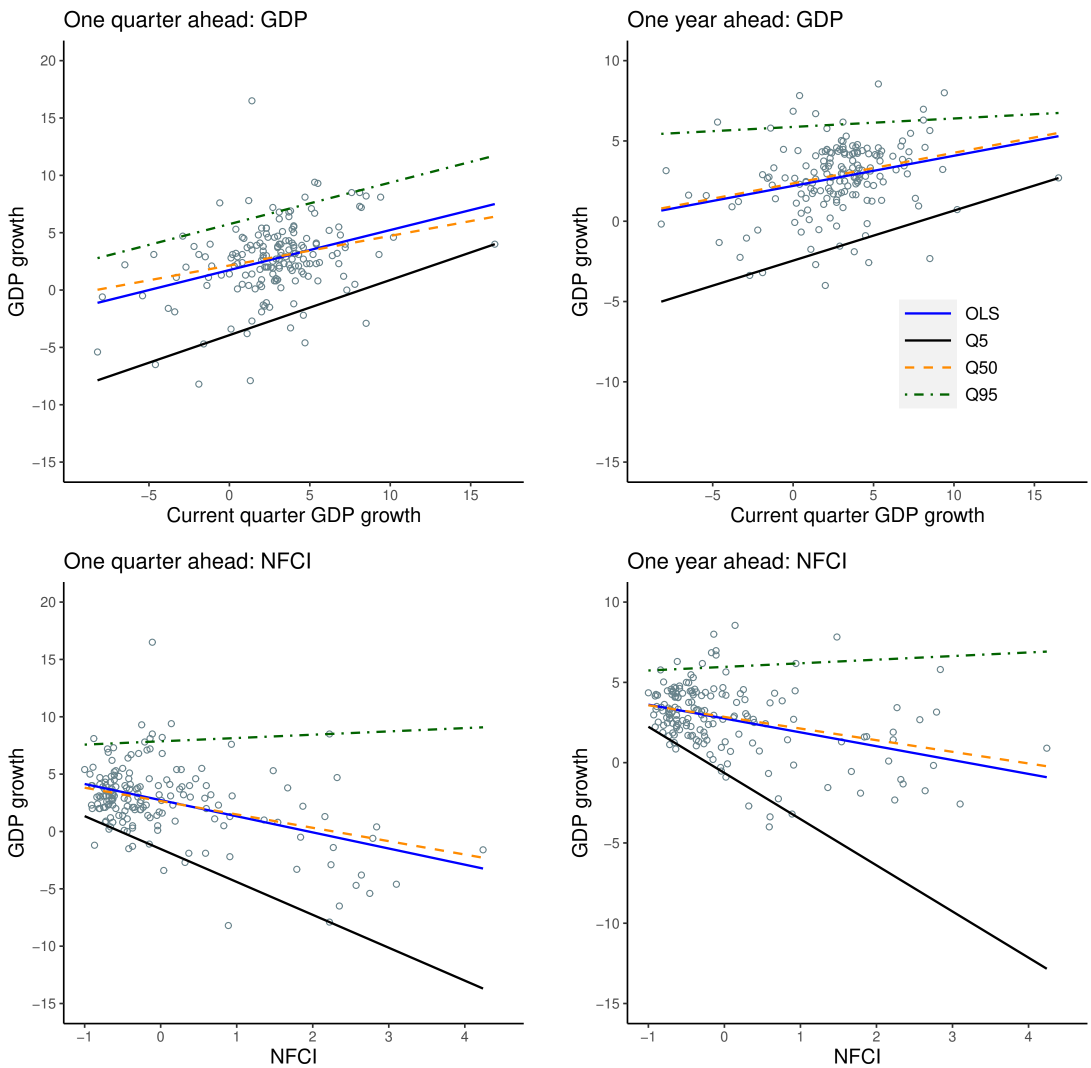
$$\hat{\beta}_\tau(h) = \underset{b_\tau(h)}{\arg\min} \sum_{t=1}^{T-h} \rho_\tau \left(y_{t+h} - x_t' b_\tau(h) \right), \quad \rho_\tau(u) = u(\tau - \mathbb{1}(u < 0)),$$

where $\mathbb{1}()$ is the indicator function.

1.1

Figure 1 plots quantile coefficients of GDP growth and NFCI based on the univariate quantile regression model. In other words, $x'_t = (1, y_t)$ or $x'_t = (1, NFCI_t)$. It replicates and is identical to Figure 3 of Adrian et al. (2019). The plot indicates that NFCI coefficients change across quantiles whereas the quantile coefficients of GDP growth do not change much across quantiles.

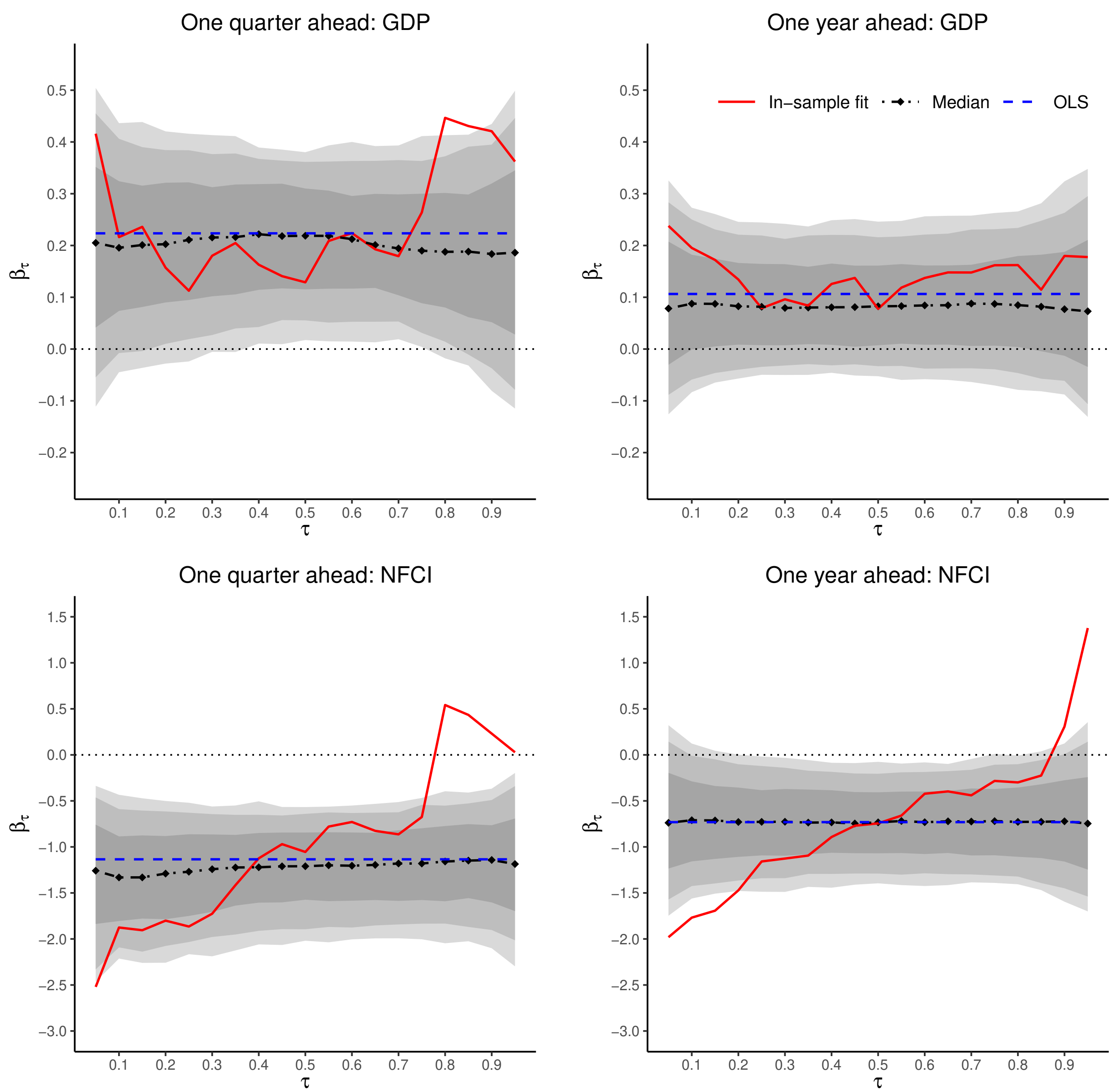
Figure 1: Replication of Figure 3 of Adrian et al. (2019)
Quantile Regression



1.2

Figure 2 below (which is Figure 1A of our paper) replicates and is almost identical to Figure 4 of Adrian et al. (2019). It plots the ordinary least squares (*OLS*) estimate and the estimated quantile coefficients (*in-sample fit*) of the quantile regression model which includes both NFCI and GDP growth as predictors, $x'_t = (1, y_t, NFCI_t)$, for $\tau \in \{0.05, 0.1, 0.2, \ldots, 0.8, 0.9, 0.95\}$. In addition, it plots the simulated 95%, 90%, and 68% confidence bands and the median based on VAR (4) with Gaussian innovations. Specifically,the confidence bands and median are computed by first fitting the linear vector autoregression process with four lags (VAR (4)), and then generating 1,000 bootstrap samples of NFCI and GDP growth of size T iteratively starting at $t = h + 1$ using the fitted VAR (4) with zero-mean homoscedastic Gaussian innovations. The variance-covariance matrix of the Gaussian distribution is calculated using the residuals of the fitted VAR (4). The confidence bands are pointwise and are obtained by simply joining the individual confidence intervals for each τ computed by percentile bootstrap method. Similarly, the median is obtained by connecting the 50% percentile of 1,000 estimated quantile coefficients for each τ.

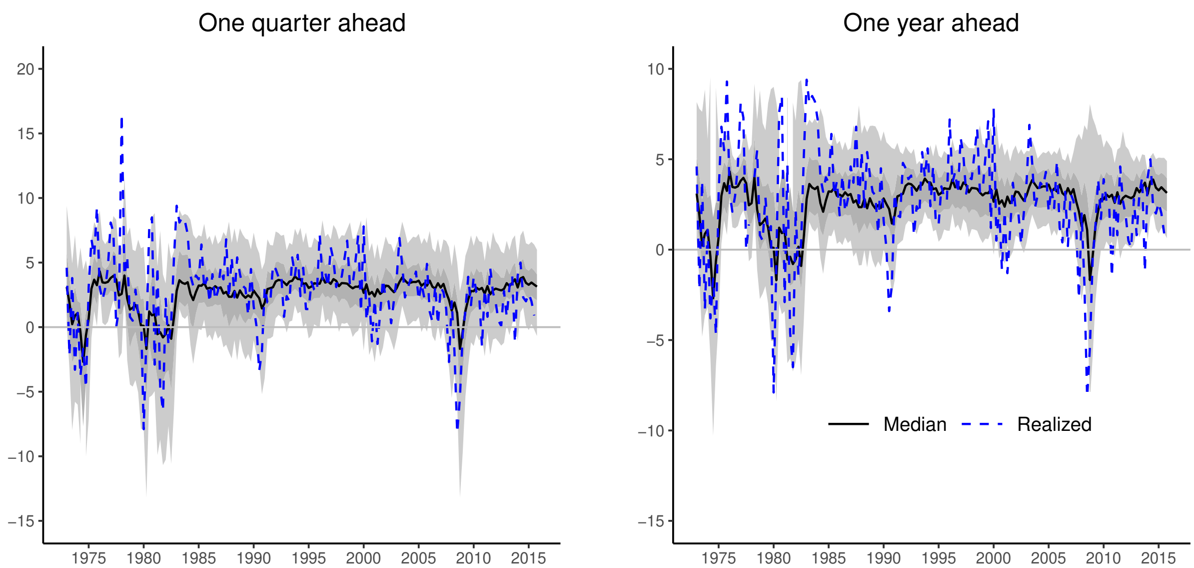
Figure 2: Replication of Figure 4 of Adrian et al. (2019)
Estimated Quantile Coefficients



1.3

Figure 3 below (which is Figure 1B of our paper) replicates Figure 5 of Adrian et al. (2019). It shows the predicted conditional quantiles for $\tau \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$ of the quantile regression model with two predictors, GDP growth and NFCI. The $\tau = 0.5$ case is denoted as *Median* and drawn in dashed line while the other quantiles are shown with bands.

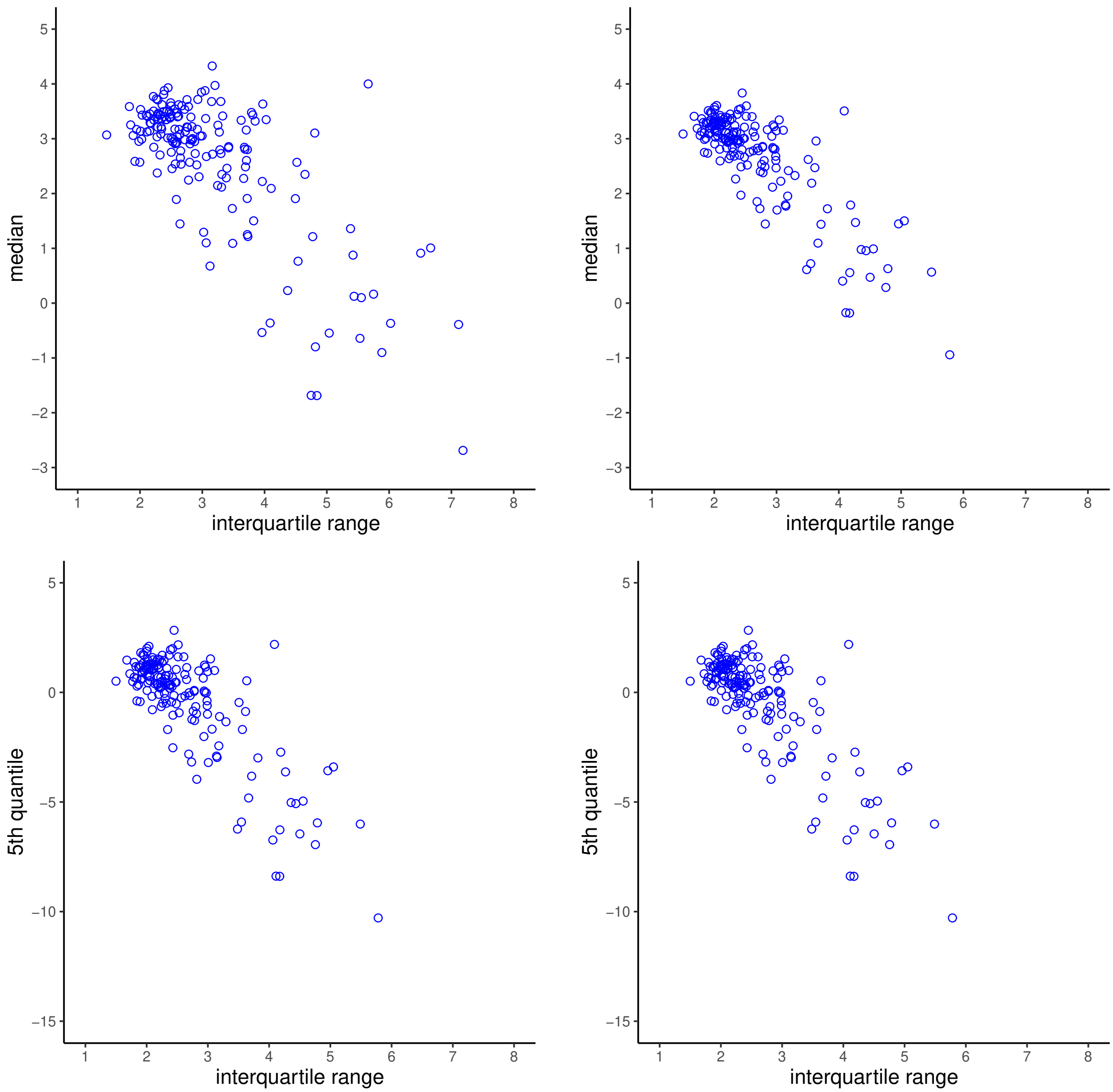
Figure 3: Replication of Figure 5 of Adrian et al. (2019)
Predicted Distribution



1.4

Figure 4 below replicates Figure 6 of Adrian et al. (2019). Figure 4 shows the predicted quantile at $\tau = 0.5$ (median) and $\tau = 0.05$ (5th quantile) on the y-axis and the estimated interquartile range, the difference between the predicted quantile at 0.75 and 0.25, on the x-axis. These predicted quantiles are estimated using the quantile regression model with two predictors, GDP growth and NFCI. Identical to Figure 6 of Adrian et al. (2019), the median and fifth quantiles are negatively correlated with the interquartile range.

Figure 4: Replicated Figure 6 of Adrian et al. (2019)
Median, IQR, and 5% Quantile of predicted Distribution

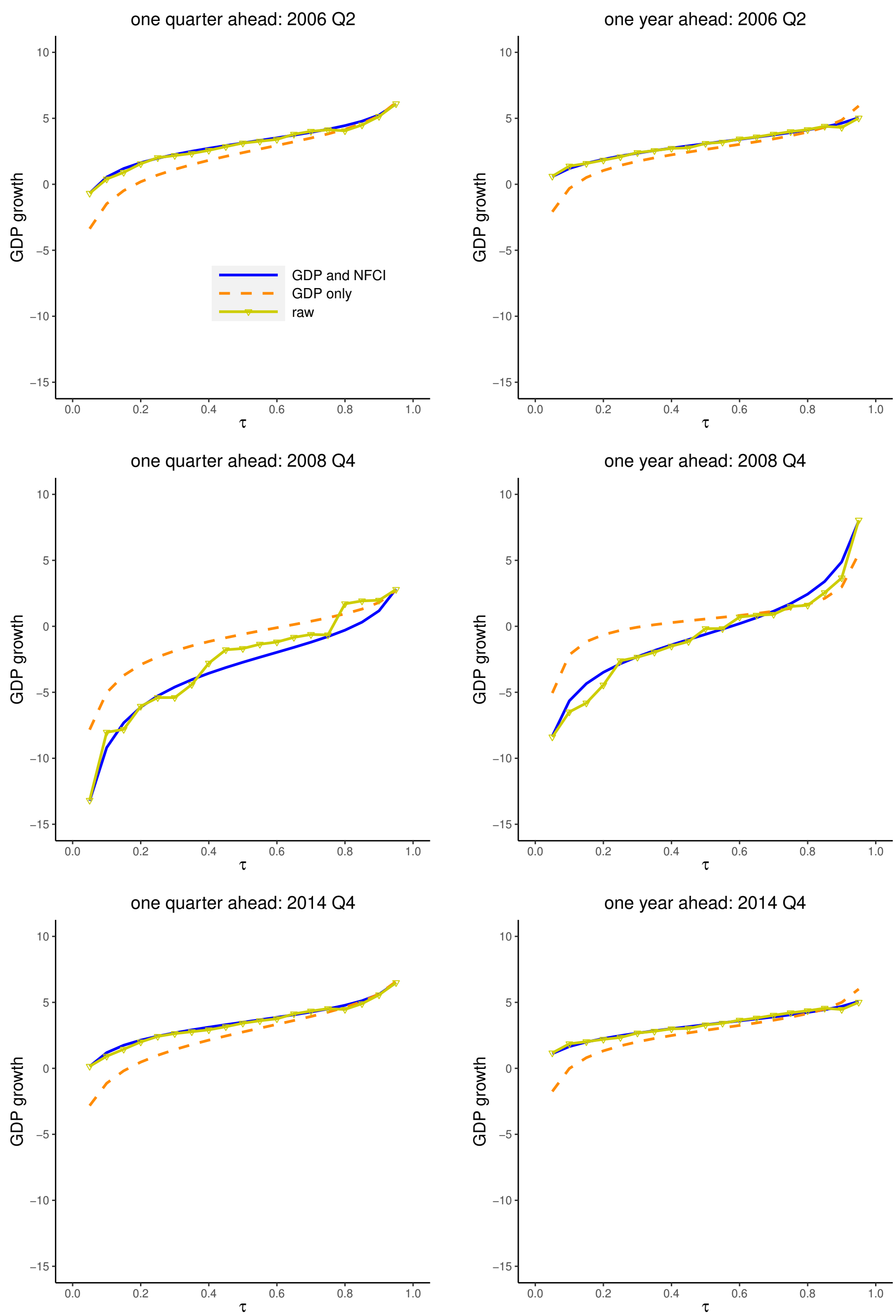


1.5

Conditional densities are estimated for each quarter using 0.05, 0.25, 0.75, and 0.95 quantiles by fitting a skewed-t distribution of Azzalini & Capitanio (2003) which depends on four parameters. Figure 5 is identical to Figure 7 of Adrian et al. (2019), which shows the conditional distribution of GDP growth for three different periods: the second quarter of 2006, the fourth quarter of 2008, and the fourth quarter of 2014. Each figure shows the conditional distributions of GDP growth. The conditional variable(s) is(are) either economic condition (GDP growth) only or both economic and financial conditions.

It can be seen that the estimated distribution only with the economic condition and the estimated distribution with both economic and financial conditions differ significantly, especially at the lower quantiles. At the upper tail, the estimated distribution considering only the economic condition is slightly above the distribution considering both economic and financial conditions. At the lower tail, the distribution that considers only economic condition is substantially below the distribution that considers both economic and financial conditions.

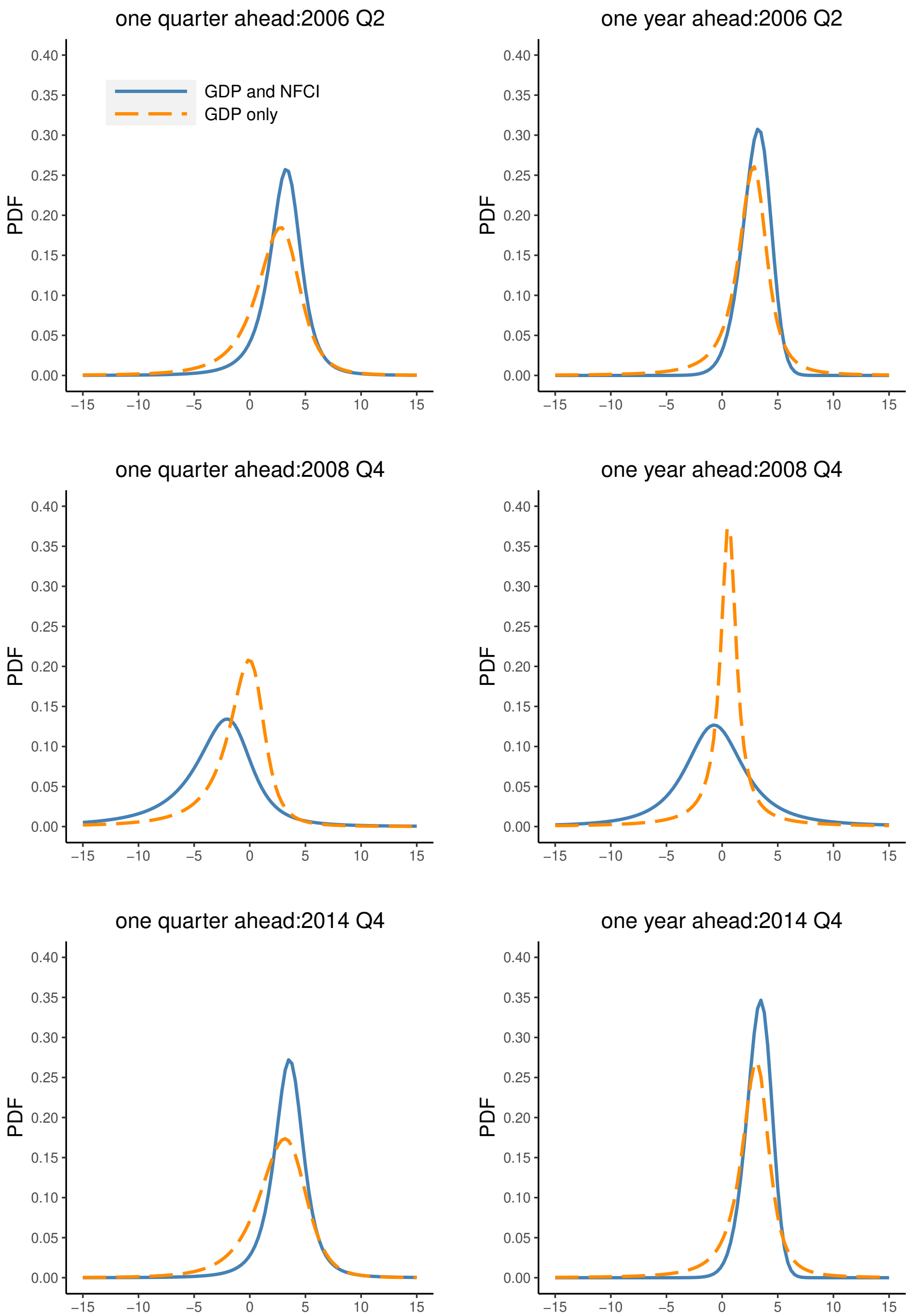
Figure 5: Replication of Figure 7 of Adrian et al. (2019)
The Conditional Quantiles and the Skewed t-Distribution



1.6

The conditional densities are estimated for each quarter using 0.05, 0.25, 0.75, and 0.95 quantiles by fitting a skewed-*t* distribution of Azzalini & Capitanio (2003) which depends on four parameters. Figure 6 is almost identical to Figure 8 of Adrian et al. (2019) and plots the fitted conditional probability density for the three periods.

Figure 6: Replication of Figure 8 of Adrian et al. (2019)



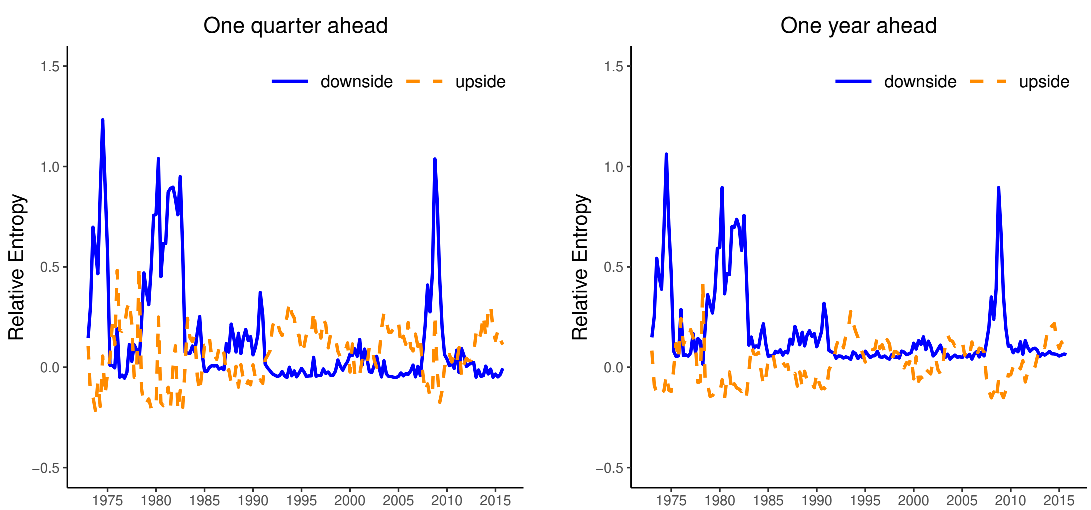
1.7

Figure 4 below (which is Figure 1C of our paper) replicates and is almost identical to Adrian et al. (2019, Figure 9 Panel A). It shows the upside (downside) risk computed by relative entropy which is the aggregated difference between the unconditional and conditional densities above (below) the median. Both the conditional (denoted as $\hat{f}()$) and unconditional (denoted as $\hat{g}()$) densities are estimated for each quarter using 0.05, 0.25, 0.75, and 0.95 quantiles by fitting a skewed-t distribution of Azzalini & Capitanio (2003) which depends on four parameters. Specifically, the following equation is used to compute the entropy.

$$\mathcal{L}^D\left(\hat{f}_{y_{t+h}|x_t};\hat{g}_{y_{t+h}}\right) = -\int_{-\infty}^{\hat{F}^{-1}_{y_{t+h}|x_t}(0.5|x_t)} \left(\log \hat{g}_{y_{t+h}}(y) - \log \hat{f}_{y_{t+h}|x_t}(y|x_t)\right)\hat{f}_{y_{t+h}|x_t}(y|x_t)dy$$
$$\mathcal{L}^U\left(\hat{f}_{y_{t+h}|x_t};\hat{g}_{y_{t+h}}\right) = -\int_{\hat{F}^{-1}_{y_{t+h}|x_t}(0.5|x_t)}^{\infty} \left(\log \hat{g}_{y_{t+h}}(y) - \log \hat{f}_{y_{t+h}|x_t}(y|x_t)\right)\hat{f}_{y_{t+h}|x_t}(y|x_t)dy.$$

Identical to Adrian et al. (2019), while both downside and upside risks fluctuate over time, downside risk moves together with NFCI and overall more pronounced than upside risk.

Figure 7: Replication of Figure 9 Panel A of Adrian et al. (2019)
Growth Entropy



2 The standard error formula used for the 90% confidence bands in Figure 2 of the paper "Reassessing Growth Vulnerability"

In this section, we provide the formula for the standard error used in Figure 2 of the paper *Reassessing Growth Vulnerability*.

2.1

Figure 8 (which is Figure 2 of our paper) reports the estimated quantile coefficients of the QR, IVX-QR, and DW-QR methods estimated through the SEE approach. The pointwise 90% confidence bands of these three estimates are computed analytically following Kaplan (2022, Section 5.5) with a Gaussian kernel and bandwidth chosen by the Silverman's rule of thumb. Specifically the standard errors are computed from the following formula.

- IVX-QR estimated with SEE:

$$\widehat{\text{var}}\left(\hat{\beta}_{1,\tau}^{IVX\text{-}QR}(h)\right) = \tau(1-\tau)\left[\frac{1}{b_k}\sum_{t=1}^{T-h}\phi\left(\frac{\hat{u}_{t+h,\tau}}{b_k}\right)x_{1,t}\tilde{z}_t'\left[\sum_{t=1}^{T-h}\tilde{z}_t\tilde{z}_t'\right]^{-1}\frac{1}{b_k}\sum_{t=1}^{T-h}\phi\left(\frac{\hat{u}_{t+h,\tau}}{b_k}\right)\tilde{z}_t x_{1,t}'\right]^{-1}$$

$$\hat{u}_{t+h,\tau} = y_{t+h,\tau} - x_{1,t}'\hat{\beta}_{1,\tau}^{IVX\text{-}QR}(h)$$

$$b_k = 1.06T^{-1/5}\left(\hat{\sigma}\wedge\frac{\widehat{IQR}}{1.34}\right)$$

$\hat{\sigma}$ and $\widehat{IQR}$ are the estimated standard deviation and IQR of $\hat{u}_{t+h,\tau}$.

- DW-QR estimated with SEE:

$$\text{Let } \widehat{\Sigma} \equiv \widehat{\text{var}}\left(\left(\begin{array}{c}\hat{\beta}_{0,\tau}(h)\\ \hat{\beta}_{1,\tau}(h)\\ \hat{\gamma}_\tau(h)\end{array}\right)\right)$$

$$= \tau(1-\tau)\left[\frac{1}{b_k}\sum_{t=1}^{T-h}\phi\left(\frac{\hat{u}_{t+h,\tau}}{b_k}\right)x_t x_t'\left[\sum_{t=1}^{T-h}x_t x_t'\right]^{-1}\frac{1}{b_k}\sum_{t=1}^{T-h}\phi\left(\frac{\hat{u}_{t+h,\tau}}{b_k}\right)x_t x_t'\right]^{-1}$$

$$x_t = \left(\begin{array}{ccc}1 & x_{1,t}^{*'} & z_t'\end{array}\right)'$$

$$\hat{u}_{t+h,\tau} = y_{t+h} - \hat{\beta}_{0,\tau}(h) - x_{1,t}^{*'}\hat{\beta}_{1,\tau}(h) - z_t'\hat{\gamma}_\tau(h)$$

Here, b_k is identical to the IVX-QR case above. Then,

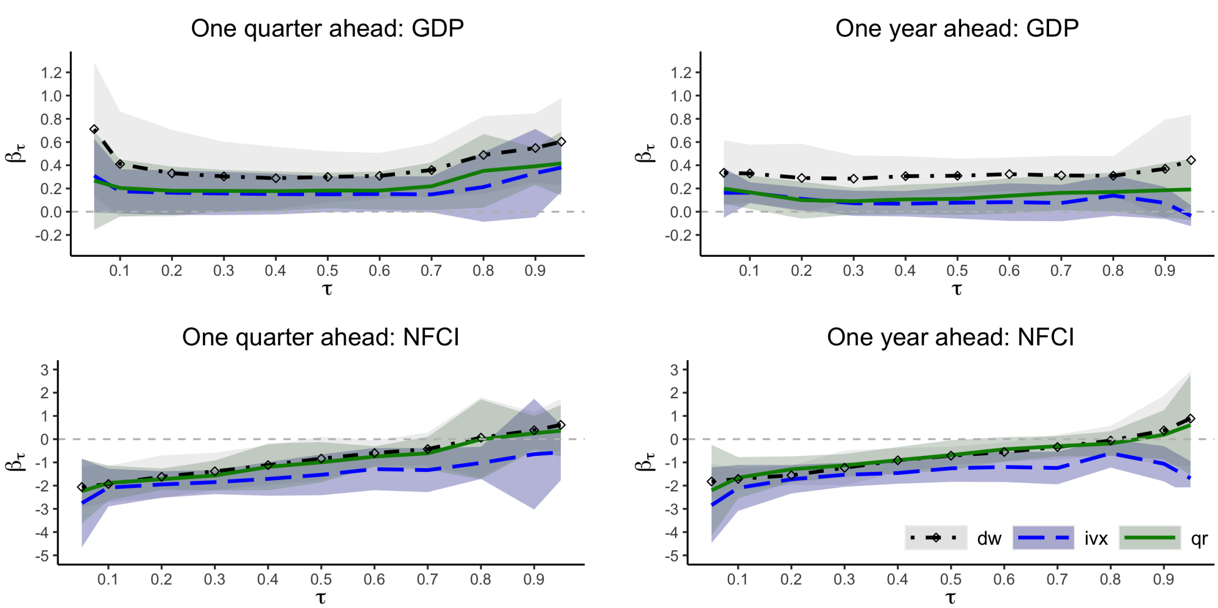
$$\hat{\beta}_{1,\tau}^{DW\text{-}QR}(h) = (W_1+W_2)^{-1}\left(W_1\hat{\beta}_{1,\tau}(h) + W_2\hat{\gamma}_\tau(h)\right)$$

$$\widehat{\text{var}}\left(\hat{\beta}_{1,\tau}^{DW\text{-}QR}(h)\right) =$$

$$\left[\begin{array}{ccc}O_{2\times1} & (W_1+W_2)^{-1}W_1 & (W_1+W_2)^{-1}W_2\end{array}\right]\widehat{\Sigma}\left[\begin{array}{ccc}O_{2\times1} & (W_1+W_2)^{-1}W_1 & (W_1+W_2)^{-1}W_2\end{array}\right]'$$

Here, $O_{2\times1}$ is a 2×1 vector of zeros.

Figure 8: Estimated Quantile Coefficients



2.2

Theorem 5 of Cai et al. (2022) and Proposition 3.1 of Lee (2016) provide results for the distribution of self-normalized test statistics based on the nonrobust standard error ($f(0)_{u_{t+h,\tau}}E[z_t x_t']$). For comparison, we also compute the standard errors based on $f(0)_{u_{t+h,\tau}}E[z_t x_t']$, and the pointwise 90% confidence bands obtained from this nonrobust standard errors are shown in Figure 9. Specifically, the nonrobust standard errors are computed based on the following formula.

- IVX-QR estimated with SEE:

$$
\begin{aligned}
\widehat{\mathrm{var}}\left(\hat{\beta}_{1,\tau}^{IVX\text{-}QR}(h)\right) &= \frac{\tau(1-\tau)}{\hat{f}_{u_\tau}(0)^2}\left[\sum_{t=1}^{T-h} x_{1,t}\tilde{z}_t'\left[\sum_{t=1}^{T-h}\tilde{z}_t\tilde{z}_t'\right]^{-1}\sum_{t=1}^{T-h}\tilde{z}_t x_{1,t}'\right]^{-1} \\
\hat{u}_{t+h,\tau} &= y_{t+h,\tau} - x_{1,t}'\hat{\beta}_{1,\tau}^{IVX\text{-}QR}(h) \\
b_k &= 1.06T^{-1/5}\left(\hat{\sigma}\wedge\frac{\widehat{IQR}}{1.34}\right) \\
\hat{f}_{u_\tau}(0) &= \frac{1}{Tb_k}\sum_{t=1}^{T-h}\phi\left(\frac{\hat{u}_{t+h,\tau}}{b_k}\right)
\end{aligned}
$$

- DW-QR estimated with SEE:

$$
\begin{aligned}
\text{Let } \widehat{\Sigma} \equiv \widehat{\mathrm{var}}\left(\begin{pmatrix}\hat{\beta}_{0,\tau}(h)\\ \hat{\beta}_{1,\tau}(h)\\ \hat{\gamma}_\tau(h)\end{pmatrix}\right) &= \frac{\tau(1-\tau)}{\hat{f}_{u_\tau}(0)^2}\left[\sum_{t=1}^{T-h} x_t x_t'\right]^{-1} \\
x_t &= \begin{pmatrix} 1 & x_{1,t}^{*'} & z_t' \end{pmatrix}' \\
\hat{u}_{t+h,\tau} &= y_{t+h} - \hat{\beta}_{0,\tau}(h) - x_{1,t}^{*'}\hat{\beta}_{1,\tau}(h) - z_t'\hat{\gamma}_\tau(h)
\end{aligned}
$$

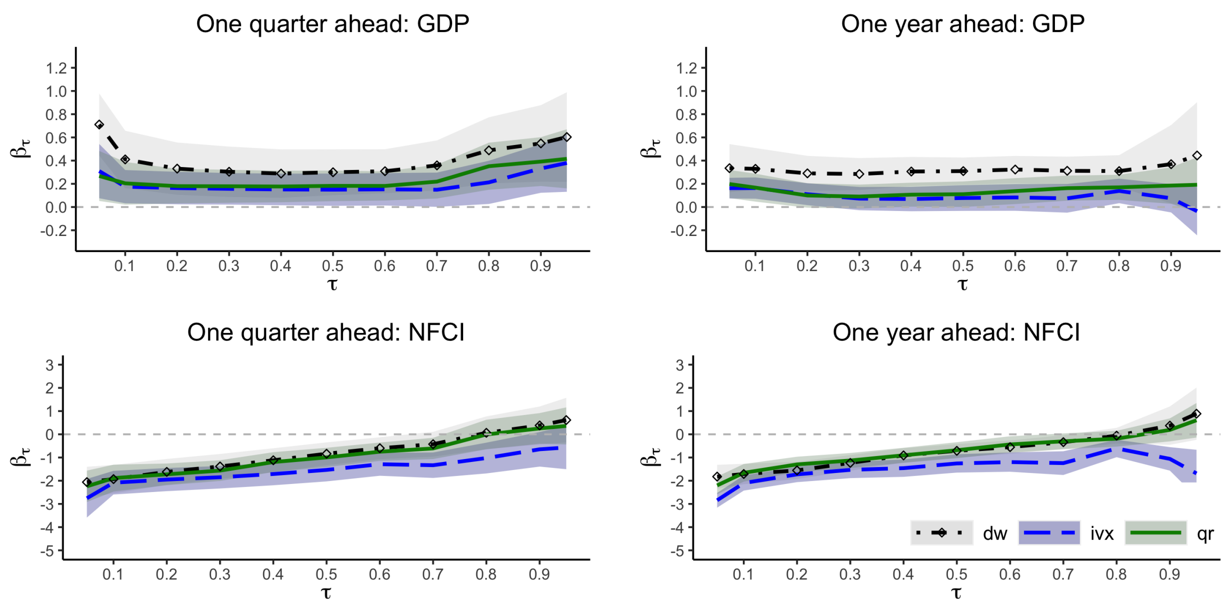
  Then,

$$
\begin{aligned}
&\widehat{\mathrm{var}}\left(\hat{\beta}_{1,\tau}^{DW\text{-}QR}(h)\right) \\
&= \begin{bmatrix} O_{2\times 1} & (W_1+W_2)^{-1}W_1 & (W_1+W_2)^{-1}W_2 \end{bmatrix}\widehat{\Sigma}\begin{bmatrix} O_{2\times 1} & (W_1+W_2)^{-1}W_1 & (W_1+W_2)^{-1}W_2 \end{bmatrix}' \\
&= \frac{\tau(1-\tau)}{\hat{f}_{u_\tau}(0)^2T^2}(W_1+W_2)^{-1}W_2(W_1+W_2)^{-1}
\end{aligned}
$$

Here $\hat{f}_{u_\tau}(0)$ is identical to the IVX-QR case.

There is a tendency for the bands to become narrower when non-robust standard errors are used compared to the bands obtained from the semi-robust standard errors as in 2.1, but the qualitative interpretation does not change.

Figure 9: Estimated Quantile Coefficients



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