Supplemental Appendix for *Inference in Time Series Models using Smoothed-Clustered Standard Errors*

Supplemental Appendix A: Additional Theoretical Results

In this supplemental appendix we provide some additional theoretical details. First we sketch the asymptotic theory for the case where the number of clusters does not evenly divide the sample. Then we sketch the calculations for the data dependent bandwidth results.

1. Clusters Do Not Evenly Divide the Sample

Suppose the last cluster has $n_\lambda < n_G$ observations. For the $G \to \infty$ with n_G fixed case, this would have asymptotically negligible impact. In the fixed-G and $n_G \to \infty$ case the last cluster matters. Assume that $n_\lambda / n_G = \lambda$ and λ is fixed as $n_G \to \infty$. The following theorem gives the limit of the CHAC statistics.

Theorem 3 *Suppose that the number of observations are not an exact multiple of G and the last cluster has n_l number of observations, $n_l < n_G$. Suppose that Assumption B is satisfied and $n_l/n_G = l$ and l is fixed as $n_G \to \infty$. Then, we have the following result.*

(a) Asymptotic normality of OLS:

$$\sqrt{T}\left(\widehat{\beta}-\beta\right)=\left(\frac{1}{T}\sum_{g=1}^{G}S_g^{xx}\right)^{-1}T^{-\frac{1}{2}}\sum_{g=1}^{G}\overline{v}_g \Rightarrow Q^{-1}\Lambda\mathcal{W}_k\left(1\right).$$

(b) CHAC result: Assume $M_G = bG$ where $b \in (0,1]$ is fixed. Define

$$P_k\left(G, M_G, \mathcal{K}\left(\cdot\right), \lambda\right) \equiv \left[\sum_{s=1}^{G-1}\sum_{h=1}^{G-1}\widetilde{\mathcal{W}}_k\left(\frac{s}{G-1+\lambda}\right)\left(2\mathcal{K}\left(\frac{|s-h|}{M_G}\right)-\mathcal{K}\left(\frac{|s-h+1|}{M_G}\right)-\mathcal{K}\left(\frac{|s-h-1|}{M_G}\right)\right)\widetilde{\mathcal{W}}_k\left(\frac{h}{G-1+\lambda}\right)'\right].$$

Then,

$$\frac{G}{T}\widehat{\Omega}^{CHAC} \equiv \Lambda P_k\left(G, M_G, \mathcal{K}\left(\cdot\right), \lambda\right)\Lambda',$$

and under H_0,

$$W_{CHAC} \Rightarrow \mathcal{W}_m(1)' P_m(G, M_G, \mathcal{K}(\cdot), \lambda)^{-1}\mathcal{W}_m(1).$$

When $m = 1$,

$$t_{CHAC} \Rightarrow \frac{\mathcal{W}_1(1)}{\sqrt{P_1(G, M_G, \mathcal{K}(\cdot), \lambda)}}.$$

Proof of Theorem 3(a): With $n_\lambda/n_G = \lambda$ and λ is fixed as $n_G \to \infty$, it follows that

$$\frac{n_G}{T} = \frac{n_G}{n_G(G-1)+n_\lambda} = \frac{1}{G-1+(n_\lambda/n_G)} = \frac{1}{G-1+\lambda}. \qquad (1)$$

Using (1), it follows that when $g \le G-1$, Assumption B2 implies that

$$\frac{1}{T}S_g^{xx} = \frac{1}{T}\sum_{t=(g-1)n_{\mathrm{G}}+1}^{gn_{\mathrm{G}}} x_t x_t' \Rightarrow \frac{g}{G-1+\lambda}Q - \frac{g-1}{G-1+\lambda}Q = \frac{1}{G-1+\lambda}Q. \tag{2}$$

When $g = G$,

$$\frac{1}{T}S_G^{xx} = \frac{1}{T}\sum_{t=(G-1)n_{\mathrm{G}}+1}^{T} x_t x_t' \Rightarrow Q - \frac{G-1}{G-1+\lambda}Q = \frac{l}{G-1+\lambda}Q. \tag{3}$$

Similarly, when $g \le G-1$, equation (1) and Assumption B3 implies

$$T^{-1/2}\overline{v}_g = \frac{1}{T}\sum_{t=(g-1)n_{\mathrm{G}}+1}^{gn_{\mathrm{G}}} v_g \Rightarrow \Lambda\left(\mathcal{W}_k\left(\frac{g}{G-1+\lambda}\right) - \mathcal{W}_k\left(\frac{g-1}{G-1+\lambda}\right)\right). \tag{4}$$

When $g = G$,

$$T^{-1/2}\overline{v}_G = \frac{1}{T}\sum_{t=(G-1)n_{\mathrm{G}}+1}^{T} v_g \Rightarrow \Lambda\left(\mathcal{W}_k(1) - \mathcal{W}_k\left(\frac{G-1}{G-1+\lambda}\right)\right). \tag{5}$$

From (2)-(5),

$$\sqrt{T}\left(\widehat{\beta}-\beta\right) = \left(\frac{1}{T}\sum_{g=1}^{G} S_g^{xx}\right)^{-1} T^{-\frac{1}{2}}\sum_{g=1}^{G}\overline{v}_g \Rightarrow Q^{-1}\Lambda\mathcal{W}_k\left(1\right).$$

□

Proof of Theorem 3(b): Define $T^{-\frac{1}{2}}\widehat{\widehat{S}}_h = \sum_{g=1}^{h}\sum_{t=(g-1)n_{\mathrm{G}}+1}^{gn_{\mathrm{G}}}\widehat{v}_t$. When $h \le G-1$,

$$\begin{aligned}
T^{-\frac{1}{2}}\widehat{\widehat{S}}_h &= \sum_{g=1}^{h} T^{-\frac{1}{2}}\overline{v}_g - \sum_{g=1}^{h}\frac{1}{T}S_g^{xx}\sqrt{T}\left(\widehat{\beta}-\beta\right) \\
&\Rightarrow \sum_{g=1}^{h}\Lambda\left[\mathcal{W}_k\left(\frac{g}{G-1+\lambda}\right) - \mathcal{W}_k\left(\frac{g-1}{G-1+\lambda}\right)\right] - \sum_{g=1}^{h}\left[\frac{g}{G-1+\lambda}Q - \frac{g-1}{G-1+\lambda}Q\right]Q^{-1}\Lambda\mathcal{W}_k\left(1\right) \\
&= \Lambda\left[\mathcal{W}_k\left(\frac{g}{G-1+\lambda}\right) - \frac{g}{G-1+\lambda}\mathcal{W}_k\left(1\right)\right] \equiv \Lambda\widetilde{\mathcal{W}}_k\left(\frac{g}{G-1+\lambda}\right).
\end{aligned}$$

The weak convergence is straightforward from (2) and (4). When $h = G$, it follows from the OLS first order conditions that $T^{-\frac{1}{2}}\widehat{\widehat{S}}_G = 0$. Using summation by parts,

$$\begin{aligned}
\frac{G}{T}\widehat{\widehat{\Omega}}^{CHAC} &= \sum_{s=1}^{G-1}\sum_{h=1}^{G-1} T^{-\frac{1}{2}}\widehat{\widehat{S}}_s\left[2\mathcal{K}\left(\frac{|s-h|}{M_G}\right) - \mathcal{K}\left(\frac{|s-h+1|}{M_G}\right) - \mathcal{K}\left(\frac{|s-h-1|}{M}\right)\right]T^{-\frac{1}{2}}\widehat{\widehat{S}}_h' \\
&\Rightarrow \Lambda\left[\sum_{s=1}^{G-1}\sum_{h=1}^{G-1}\widetilde{\mathcal{W}}_k\left(\frac{s}{G-1+\lambda}\right)\left(2\mathcal{K}\left(\frac{|s-h|}{M_G}\right) - \mathcal{K}\left(\frac{|s-h+1|}{M_G}\right) - \mathcal{K}\left(\frac{|s-h-1|}{M_G}\right)\right)\widetilde{\mathcal{W}}_k\left(\frac{h}{G-1+\lambda}\right)'\right]\Lambda' \\
&\equiv \Lambda P_k\left(G, M_G, \mathcal{K}\left(\cdot\right), \lambda\right)\Lambda'.
\end{aligned}$$

Then, it is straightforward to show that

$$
\begin{aligned}
W_{CHAC} &= \left(R\widehat{\beta} - r\right)' \left[R\widehat{V}_{CHAC}R'\right]^{-1} \left(R\widehat{\beta} - r\right) \\
&= \sqrt{T}\left(R\widehat{\beta} - r\right)' \left[RT\widehat{V}_{CHAC}R'\right]^{-1} \sqrt{T}\left(R\widehat{\beta} - r\right) \\
&\Rightarrow \left[RQ^{-1}\Lambda\mathcal{W}_k(1)\right]' \left[RQ^{-1}\Lambda P_k(G, M_G, \mathcal{K}(\cdot), \lambda)\Lambda'^{-1}R\right]^{-1} RQ^{-1}\Lambda\mathcal{W}_k(1) \\
&= \mathcal{W}_m(1)' P_m(G, M_G, \mathcal{K}(\cdot), \lambda)^{-1}\mathcal{W}_m(1).
\end{aligned}
$$

When $m = 1$,

$$
t_{CHAC} = \frac{R\widehat{\beta} - r}{\sqrt{R\widehat{V}_{CHAC}R'}} \Rightarrow \frac{\mathcal{W}_1(1)}{\sqrt{P_1(G, M_G, \mathcal{K}(\cdot), \lambda)}}.
$$

□

2. Data Dependent Bandwidth Formulas

This section sketches the derivation of the data dependent bandwidth results given in Section 5. We first derive formulas for the MSE-optimal bandwidth followed by the test-optimal bandwidth. We begin by stating the well known result that if v_t is an AR(1) process then $\overline{v}_g$ is an ARMA(1,1) process and the AR and MA parameters are functions of the parameters of the original AR(1) process, v_t. The proof follows Amemiya and Wu (1972) and is omitted.

Result 1 *Let v_t be an AR(1) process with an AR coefficient ρ:*

$$
v_t = \rho v_{t-1} + \varepsilon_t, \quad var(\varepsilon_t) = \sigma_\varepsilon^2.
$$

Then, the non-overlapping time aggregated process with n_G time periods, $\overline{v}_g = \sum_{t=(g-1)n_G+1}^{gn_G} v_t$, is an ARMA(1,1) process

$$
\overline{v}_g = \phi\overline{v}_{g-1} + e_g + \eta e_{g-1}, \; g = 1, \ldots, G,
$$

with AR and MA coefficients that are functions of ρ, n_G, and σ_ε^2 given by

$$
\phi = \rho^{n_G}, \quad \eta = \frac{2\gamma_1^*}{\gamma_0^* + \sqrt{\gamma_0^{*2} - 4\gamma_1^{*2}}},
$$

where

$$
\begin{aligned}
\sigma_e^2 &= \frac{\gamma_0^* + \sqrt{\gamma_0^{*2} - 4\gamma_1^{*2}}}{2}, \\
\gamma_0^* &= \left[\sum_{j=0}^{n_G-1}\left(\sum_{i=0}^{j}\rho^i\right)^2 + \sum_{j=0}^{n_G-2}\left(\sum_{i=0}^{j}\rho^{n_G-1-i}\right)^2\right]\sigma_\varepsilon^2, \\
\gamma_1^* &= \sum_{j=1}^{n_G-1}\left[\left(\sum_{i=j+1}^{n_G}\rho^{i-1}\right)\left(\sum_{i=0}^{j-1}\rho^i\right)\right]\sigma_\varepsilon^2.
\end{aligned}
$$

Using Result 1 the following holds for the long run variances and derivatives of the spectral densities evaluated at frequency zero of v_t and $\overline{v}_g$.

Result 2 *Let v_t be an AR(1) process with an AR coefficient ρ:*

$$v_t = \rho v_{t-1} + \varepsilon_t, \quad var(\varepsilon_t) = \sigma_\varepsilon^2.$$

Define the non-overlapping time aggregated process, $\overline{v}_g = \sum_{t=(g-1)n_G+1}^{gn_G} v_t$. Let $\Omega^{(q)} = \sum_{j=-\infty}^{\infty} |j|^q \Gamma_j$ and $\Omega_c^{(q)} = \sum_{j=-\infty}^{\infty} |j|^q \Gamma_{cj}$, where Γ_j and Γ_{cj} are the autocovariance functions of v_t and $\overline{v}_g$, respectively. Then, the following equalities hold:

$$\begin{aligned} \Omega_c &= n_G \Omega, \\ \Omega_c^{(1)} &= \Omega^{(1)}, \\ \Omega_c^{(2)} &= \Omega^{(2)} \frac{(1+\rho^{n_G})(1-\rho)}{(1-\rho^{n_G})(1+\rho)}. \end{aligned}$$

Proof of Result 2: From Result 1, $\overline{v}_g$ is the ARMA(1,1) process

$$(1-\phi L)\overline{v}_g = (1-\eta L)e_g, \ g = 1, \ldots,$$

with ϕ, η, σ_e^2 defined as functions of ρ and σ_ε^2. The autocovariance functions is given by

$$\Gamma_{c0} = \sigma_e^2 \left(1 + \frac{(\phi+\eta)^2}{1-\phi^2} \right), \quad \Gamma_{c1} = \sigma_e^2 \frac{(\phi+\eta)(1+\phi\eta)}{1-\phi^2}, \quad \Gamma_{cj} = \phi \Gamma_{c\,j-1}, j \geq 2,$$

and straightforward calculations give

$$\begin{aligned} \Omega_c &= \sum_{j=-\infty}^{\infty} \Gamma_{cj} = \sigma_e^2 \left(\frac{1+\eta}{1-\phi} \right)^2, \\ \Omega_c^{(1)} &= \sum_{j=-\infty}^{\infty} |j| \Gamma_{cj} = 2\sigma_e^2 \frac{(\phi+\eta)(1+\phi\eta)}{(1-\phi)^3(1+\phi)}, \\ \Omega_c^{(2)} &= \sum_{j=-\infty}^{\infty} j^2 \Gamma_{cj} = 2\sigma_e^2 \frac{(\phi+\eta)(1+\phi\eta)}{(1-\phi)^4}. \end{aligned}$$

First note that using $\phi = \rho^{n_G}$, $\gamma_0^* = (1+\eta^2)\sigma_e^2$, and $\gamma_1^* = \eta\sigma_e^2$ in Result 1 we have the following:

$$(1+\eta)^2\sigma_e^2 = (1+\eta^2)\sigma_e^2 + 2\eta\sigma_e^2 = \gamma_0^* + 2\gamma_1^* = \frac{n_G(1-\rho^{n_G})^2}{(1-\rho)^2}\sigma_\varepsilon^2$$

and

$$\begin{aligned} (\phi+\eta)(1+\phi\eta)\sigma_e^2 &= \phi(1+\eta^2)\sigma_e^2 + (1+\phi^2)\eta\sigma_e^2 \\ &= \left(\rho^{n_G}\gamma_0^* + \left(1+\rho^{2n_G}\right)\gamma_1^*\right)\sigma_\varepsilon^2 \\ &= \frac{\rho\left(1-\rho^{n_G}\right)^3\left(1+\rho^{n_G}\right)}{(1-\rho)^3(1+\rho)}\sigma_\varepsilon^2. \end{aligned}$$

Plugging these expressions into Ω_c, $\Omega_c^{(1)}$, and $\Omega_c^{(2)}$ gives

$$\Omega_c = \frac{n_{\mathrm{G}}}{(1-\rho)^2}\sigma_\varepsilon^2 = n_{\mathrm{G}}\Omega,$$

$$\Omega_c^{(1)} = 2\frac{\frac{\rho(1-\rho^{n_{\mathrm{G}}})^3(1+\rho^{n_{\mathrm{G}}})}{(1-\rho)^3(1+\rho)}}{(1-\rho^{n_{\mathrm{G}}})^3(1+\rho^{n_{\mathrm{G}}})}\sigma_\varepsilon^2 = \frac{2\rho}{(1-\rho)^3(1+\rho)}\sigma_\varepsilon^2 = \Omega^{(1)},$$

$$\Omega_c^{(2)} = 2\frac{\frac{\rho(1-\rho^{n_{\mathrm{G}}})^3(1+\rho^{n_{\mathrm{G}}})}{(1-\rho)^3(1+\rho)}}{(1-\rho^{n_{\mathrm{G}}})^4}\sigma_\varepsilon^2 = \frac{2\rho(1+\rho^{n_{\mathrm{G}}})}{(1-\rho)^3(1+\rho)(1-\rho^{n_{\mathrm{G}}})}\sigma_\varepsilon^2 = \Omega^{(2)}\frac{(1+\rho^{n_{\mathrm{G}}})(1-\rho)}{(1-\rho^{n_{\mathrm{G}}})(1+\rho)}.$$

□

Result 3 *Denote the MSE-optimal bandwidth without clustering as* M_T^* *and the MSE-optimal* (M_G, n_G) *pair as* (M_G^*, n_G^*). *Suppose that* v_t *is an AR(1) process with AR parameter* ρ. *Then, the following holds.*

1. *For kernels with* $q = 1$, *the minimization of the CHAC-MSE can only determine the product* ${n_G}^* M_G^*$ *but not* ${n_G}^*$ *and* M_G^* *individually and the following equality holds:*

$$n_G^* M_G^* = \left(\frac{k_1^2}{c_1}\left(\frac{\Omega^{(1)}}{\Omega^2}\right)^2 T\right)^{\frac{1}{3}} = M_T^*.$$

2. *For kernels with* $q = 2$ *suppose that* $\widehat{\Omega}^{(2)} > 0$ *and* $\rho > 0$. *Then the minimization of CHAC-MSE has a corner solution with* $n_G^* = 1$.

Proof of Result 3: The notation used in this proof is defined in Section 5. Following Andrews (1991), the MSE of the usual HAC estimator is

$$MSE\left(\widehat{\Omega}\right) \approx \left(\frac{k_q\Omega^{(q)}}{M_T}\right)^2 + 2c_2\Omega^2\frac{M_T}{T},$$

where M_T is the bandwidth, $q \in [0, \infty)$ is the largest integer such that $k_q = lim_{x\to 0}\frac{1-\mathcal{K}(x)}{|x|^q} < \infty$, and $c_2 = \int \mathcal{K}(x)^2 dx$. Similarly, for the CHAC estimator, when $G \to \infty$,

$$MSE\left(\frac{1}{n_{\mathrm{G}}}\widehat{\Omega}^{CHAC}\right) = \frac{1}{n_{\mathrm{G}}^2}MSE\left(\widehat{\Omega}^{CHAC}\right) \approx \frac{1}{n_{\mathrm{G}}^2}\left[\left(\frac{k_q\Omega_c^{(q)}}{M_G}\right)^2 + 2c_2\Omega_c^2\frac{M_G}{G}\right],$$

where M_G is the bandwidth. When v_t is an AR(1) process, using Results 1 and 2 we can rewrite Ω_c and $\Omega_c^{(q)}$ as functions of Ω and $\Omega^{(q)}$. With $T = n_{\mathrm{G}}G$, the MSE criteria for $q = 1$ (Bartlett) and $q = 2$ kernels become

$$MSE\left(\frac{1}{n_{\mathrm{G}}}\widehat{\Omega}^{CHAC}\right) = \begin{cases} \left(\frac{k_1\Omega^{(1)}}{n_{\mathrm{G}}M_G}\right)^2 + 2c_1\Omega^2\frac{n_{\mathrm{G}}M_G}{T} & q = 1 \\ \left(\frac{k_2\Omega^{(2)}}{n_{\mathrm{G}}M_G^2}\frac{(1+\rho^{n_{\mathrm{G}}})(1-\rho)}{(1-\rho^{n_{\mathrm{G}}})(1+\rho)}\right)^2 + 2c_2\Omega^2\frac{n_{\mathrm{G}}M_G}{T} & q = 2. \end{cases}$$

For $q = 1$ the MSE formula depends on n_{G} and M_G only through the product $n_{\mathrm{G}} M_G$. Therefore, minimization of the MSE can only determine the product but not n_{G} and M_G individually. If we replace $n_{\mathrm{G}} M_G$ with M_T, then the MSE criteria is identical to the no-clustering case. By straightforward calculation, we have the following equality:

$$n_{\mathrm{G}}^* M_G^* = M_T^* = \left(\frac{k_1^2}{c_1} \left(\frac{\Omega^{(1)}}{\Omega^2} \right)^2 T \right)^{\frac{1}{3}}.$$

This expression can be further simplified with $k_1 = 1$ and $c_2 = 2/3$ for the Bartlett kernel.

For the kernels with $q = 2$, to obtain the bandwidth and the size of the cluster that jointly minimize the MSE criterion, first take the cluster size, n_{G}, as given. Suppose that $\Omega^{(2)} > 0$. Then by straightforward computation,

$$\frac{\partial MSE}{\partial M_G} = \left(k_2 \Omega^{(2)} \frac{(1+\rho^{n_{\mathrm{G}}})(1-\rho)}{(1-\rho^{n_{\mathrm{G}}})(1+\rho)} \right)^2 \frac{-4}{n_{\mathrm{G}}^2 M_G^5} + \frac{n_{\mathrm{G}} 2 c_2 \Omega^2}{T} = 0.$$

Solving for M_G gives

$$M_G^* = \left[\frac{2T \left(k_2 \Omega^{(2)}\right)^2}{c_2 \Omega^2} \left(\frac{(1+\rho^{n_{\mathrm{G}}})(1-\rho)}{(1-\rho^{n_{\mathrm{G}}})(1+\rho)} \right)^2 \frac{1}{n_{\mathrm{G}}^3} \right]^{1/5} = M_T^* \left[\left(\frac{(1+\rho^{n_{\mathrm{G}}})(1-\rho)}{(1-\rho^{n_{\mathrm{G}}})(1+\rho)} \right)^2 \frac{1}{n_{\mathrm{G}}^3} \right]^{1/5}.$$

Time series with positive serial correlation satisfy $\Omega^{(2)} > 0$. Plugging M_G^* back in to the CHAC-MSE criterion, the concentrated MSE criterion function, denoted by $MSE(M_G^*)$, becomes

$$\begin{aligned} MSE(M_G^*) &= \left[\left(k_2 \Omega^{(2)}\right)^2 + \frac{2 c_2 \Omega^2}{T} \right] \left(\frac{(1+\rho^{n_{\mathrm{G}}})(1-\rho)}{(1-\rho^{n_{\mathrm{G}}})(1+\rho)} \right)^{2/5} n_{\mathrm{G}}^{2/5} \\ &= n_{\mathrm{G}}^{2/5} \left(\frac{(1+\rho^{n_{\mathrm{G}}})}{(1-\rho^{n_{\mathrm{G}}})} \right)^{2/5} \mathcal{C}. \end{aligned}$$

Here $\mathcal{C}$ is a positive constant that does not depend on n_{G}. This expression is increasing in n_{G} when $0 < \rho < 1$. Therefore, the MSE minimization has a corner solution with $n_G^* = 1$ when $\rho > 0$. □

Similar results hold for the test-optimal bandwidth approach as given by the following result.

Result 4 *Denote the test-optimal bandwidth without clustering as M_T^* and the test-optimal (bandwidth, size of a cluster) as (M_G^*, n_G^*) with clustering. Suppose that v_t is an AR(1) process with the AR coefficient ρ. Then, we have the following results.*

1. *Suppose that $(\Omega^{(1)}/\Omega) \left\{ w G_{1,0}' \left(z^2\right) - G_{1,\delta}' \left(z^2\right) \right\} > 0$. Then, for the kernels with $q = 1$, the minimization of the CHAC-SPJ loss function can only determine the product $n_G^* M_G^*$ but not $n_G{}^*$ and M_G^* individually and the following equality holds:*

$$n_G^* M_G^* = M_T^*.$$

2. *Suppose that $(\Omega^{(2)}/\Omega) \left\{ w G_{1,0}' \left(z^2\right) - G_{1,\delta}' \left(z^2\right) \right\} > 0$. Then, for kernels with $q = 2$, the minimization of CHAC-SPJ loss function has a corner solution with $n_G^* = 1$.*

Proof of Result 4 Following Sun, Phillips and Jin (2008) (SPJ), the test-optimal bandwidth minimizes the SPJ objective function, which is a weighted average of the approximate type I and the type II errors of the test statistic. With weight $w/(w+1)$ on the type I error and a fixed local alternative, the loss function for the usual HAC approach (no clustering) is given by

$$\mathcal{L}(M;\delta,T,z) = k_q \frac{\Omega^{(q)}}{\Omega}\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} z^2 (M_T)^{-q} + c_2 z^4 \mathbb{K}_\delta(z^2)\frac{M_T}{T}$$

after dropping a term which does not depend on M and scaling by $(1+w)$. Here, $G_{q,\lambda}()$ is the cdf of a non-central chi-square-q random variable with non-centrality parameter λ^2, $\mathbb{K}_\delta(x) = \delta^2 G'_{3,\delta}(x)/2x$, and δ is a parameter that defines the alternative hypothesis (see Sun et al. (2008) for details). Similarly, for the CHAC approach, the SPJ objective function is

$$\mathcal{L}^{CHAC}(M_G, n_{\mathrm{G}};\delta,G,z) = k_q \frac{\Omega_c^{(q)}}{\Omega_c}\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} z^2 (M_G)^{-q} + c_2 z^4 \mathbb{K}_\delta(z^2)\frac{M_G,}{G}.$$

When v_t is an AR(1) process, $\overline{v}_g$ is an ARMA(1,1) process (Result 1). Using Result 2, $\Omega_c^{(q)}$ and Ω_c can be rewritten in terms of $\Omega^{(q)}$ and Ω. Then, with $T = n_{\mathrm{G}}G$, the SPJ loss function becomes

$$\begin{aligned}&\mathcal{L}^{CHAC}(M_G, n_{\mathrm{G}};\delta,T,z)\\ =&\begin{cases} k_1 \frac{\Omega^{(1)}}{\Omega}\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} z^2 (n_{\mathrm{G}}M_G)^{-1} + c_2 z^4 \mathbb{K}_\delta(z^2)\frac{M_G n_{\mathrm{G}}}{T} & q=1\\ k_2 \frac{\Omega^{(2)}}{\Omega}\left(n_{\mathrm{G}}\frac{1+\rho^{n_{\mathrm{G}}}}{1-\rho^{n_{\mathrm{G}}}}\frac{1-\rho}{1+\rho}\right)\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} z^2 (M_G n_{\mathrm{G}})^{-2} + c_2 z^4 \mathbb{K}_\delta(z^2)\frac{M_G n_{\mathrm{G}}}{T} & q=2\end{cases}.\end{aligned}$$

For the $q=1$ kernels (Bartlett kernel), note that the SPJ loss function depends on n_{G} and M_G only through the product $n_{\mathrm{G}}M_G$. Therefore, minimization of the SPJ loss function can only determine the product but not n_{G} and M_G individually. If we replace $n_{\mathrm{G}}M_G$ with M_T, then the SPJ loss functions is the same as the no-clustering case. Therefore, by straightforward calculation we have the following equality:

$$n_{\mathrm{G}}^* M_G^* = M_T^* = \left(\frac{k_1 z^2}{c_2 z^4 \mathbb{K}_\delta(z^2)}\frac{\Omega^{(1)}}{\Omega}\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} T\right)^{\frac{1}{2}}.$$

This expression can be further simplified with $k_1 = 1$ and $c_2 = 2/3$ for the Bartlett kernel.

Next, consider the kernels with $q=2$. Let n_{G} be given. By straightforward calculation,

$$\begin{aligned}&\frac{\partial\mathcal{L}^{CHAC}(M_G;n_{\mathrm{G}},\delta,T,z)}{\partial M_G}\\ =&k_2\frac{\Omega^{(2)}}{\Omega}\left(n_{\mathrm{G}}\frac{1+\rho^{n_{\mathrm{G}}}}{1-\rho^{n_{\mathrm{G}}}}\frac{1-\rho}{1+\rho}\right)\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} z^2 (n_{\mathrm{G}})^{-2} M_G^{-3}(-2) + c_2 z^4 \mathbb{K}_\delta(z^2)\frac{n_{\mathrm{G}}}{T} = 0.\end{aligned}$$

When $(\Omega^{(q)}/\Omega)\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} > 0$, the test-optimal M_G^*, given n_{G}, is

$$M_G^* = \left(\frac{2k_2\frac{\Omega^{(2)}}{\Omega}\left(\frac{1+\rho^{n_{\mathrm{G}}}}{1-\rho^{n_{\mathrm{G}}}}\frac{1-\rho}{1+\rho}\right)\left\{wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right)\right\} z^2 T}{c_2 z^4 \mathbb{K}_\delta(z^2) n_{\mathrm{G}}^2}\right)^{1/3} = M_T^*\left(\frac{1}{n_{\mathrm{G}}^2}\frac{1+\rho^{n_{\mathrm{G}}}}{1-\rho^{n_{\mathrm{G}}}}\frac{1-\rho}{1+\rho}\right)^{1/3}.$$

Using this formula for M_G^*, the concentrated loss function, denoted by $\mathcal{L}^{CHAC}(n_{\mathrm{G}}; M_G^*, \delta, T, z)$, is

$$\begin{aligned}
&\mathcal{L}^{CHAC}(n_{\mathrm{G}}; M_G^*, \delta, T, z) \\
&= \left\{ k_2 \frac{\Omega^{(2)}}{\Omega} \left(\frac{1+\rho^{n_{\mathrm{G}}}}{1-\rho^{n_{\mathrm{G}}}} \frac{1-\rho}{1+\rho} \right) \left\{ wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right) \right\} z^2 n_{\mathrm{G}} \left(\frac{c_2 z^4 \mathbb{K}_\delta(z^2)}{T} \right)^2 \right\}^{1/3} \left(2^{\frac{-2}{3}} + 2^{\frac{1}{3}} \right) \\
&= \left(n_{\mathrm{G}} \frac{1+\rho^{n_{\mathrm{G}}}}{1-\rho^{n_{\mathrm{G}}}} \right)^{1/3} \mathcal{C},
\end{aligned}$$

where $\mathcal{C}$ is a constant that does not depend on n_{G} and is positive if $(\Omega^{(2)}/\Omega)\left\{ wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right) \right\} > 0$. The expression, $\left(n_{\mathrm{G}} \frac{1+\rho^{n_{\mathrm{G}}}}{1-\rho^{n_{\mathrm{G}}}} \right)$, is an increasing function of n_{G}. Hence, minimization of the SPJ loss function has a corner solution at $n_G^* = 1$ if $(\Omega^{(2)}/\Omega)\left\{ wG'_{1,0}\left(z^2\right) - G'_{1,\delta}\left(z^2\right) \right\} > 0$. □

Supplemental Appendix B: Asymptotic Critical Values

This section reports simulated asymptoptic null critical values for the Bartlett kernel t_{CHAC} statistic using $n_G \to \infty$ and G-fixed asymptotics (and hence M_G/G is fixed as well) as in Theorem 2.

Table B: Fixed-G, large-n_G Asymptotic Critical Values

G	M_G	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
2	1	−45.991	−17.920	−8.992	−4.390	−0.010	4.375	8.874	17.942	46.230
2	2	−65.041	−25.342	−12.716	−6.208	−0.014	6.187	12.550	25.374	65.379
3	1	−8.710	−5.323	−3.605	−2.325	−0.008	2.305	3.563	5.227	8.680
3	2	−11.315	−7.057	−4.702	−2.997	−0.009	2.980	4.618	6.805	11.286
3	3	−13.858	−8.642	−5.759	−3.671	−0.012	3.650	5.656	8.334	13.823
4	1	−5.303	−3.670	−2.724	−1.917	−0.008	1.896	2.723	3.676	5.214
4	2	−6.945	−4.716	−3.428	−2.349	−0.008	2.346	3.409	4.679	6.769
4	3	−8.005	−5.603	−4.038	−2.782	−0.010	2.764	4.045	5.518	7.931
4	4	−9.243	−6.470	−4.663	−3.212	−0.012	3.191	4.671	6.371	9.158
5	1	−4.143	−3.120	−2.407	−1.732	−0.007	1.720	2.381	3.124	4.240
5	2	−5.272	−3.857	−2.907	−2.056	−0.008	2.050	2.886	3.829	5.322
5	3	−6.288	−4.540	−3.407	−2.403	−0.009	2.390	3.397	4.492	6.246
5	4	−7.010	−5.136	−3.874	−2.720	−0.010	2.710	3.847	5.092	7.069
5	5	−7.837	−5.742	−4.331	−3.041	−0.011	3.029	4.301	5.693	7.903
6	1	−3.693	−2.837	−2.230	−1.628	−0.007	1.623	2.209	2.805	3.641
6	2	−4.526	−3.396	−2.615	−1.887	−0.007	1.872	2.598	3.349	4.514
6	3	−5.356	−3.980	−3.022	−2.159	−0.008	2.152	3.026	3.915	5.301
6	4	−6.006	−4.507	−3.430	−2.434	−0.009	2.410	3.400	4.427	5.945
6	5	−6.619	−4.942	−3.775	−2.684	−0.010	2.671	3.754	4.883	6.564
6	6	−7.251	−5.414	−4.136	−2.940	−0.011	2.926	4.112	5.349	7.190
7	1	−3.405	−2.658	−2.114	−1.569	−0.006	1.554	2.108	2.651	3.401
7	2	−4.057	−3.114	−2.431	−1.778	−0.007	1.768	2.413	3.089	4.110
7	3	−4.752	−3.576	−2.779	−2.004	−0.008	1.992	2.764	3.570	4.770
7	4	−5.401	−4.032	−3.128	−2.239	−0.009	2.225	3.099	4.003	5.358
7	5	−5.949	−4.436	−3.461	−2.470	−0.009	2.448	3.409	4.436	5.913
7	6	−6.394	−4.823	−3.749	−2.681	−0.010	2.666	3.709	4.816	6.358
7	7	−6.906	−5.209	−4.050	−2.896	−0.011	2.879	4.006	5.202	6.868
8	1	−3.210	−2.526	−2.048	−1.522	−0.006	1.516	2.035	2.530	3.208
8	2	−3.749	−2.915	−2.311	−1.704	−0.007	1.693	2.298	2.908	3.746
8	3	−4.346	−3.324	−2.611	−1.899	−0.007	1.895	2.593	3.324	4.351
8	4	−4.945	−3.739	−2.907	−2.100	−0.008	2.095	2.893	3.720	4.878
8	5	−5.457	−4.109	−3.205	−2.300	−0.009	2.289	3.185	4.082	5.364
8	6	−5.876	−4.454	−3.474	−2.485	−0.009	2.484	3.452	4.445	5.799
8	7	−6.279	−4.788	−3.731	−2.673	−0.010	2.675	3.698	4.767	6.228
8	8	−6.712	−5.118	−3.989	−2.857	−0.011	2.860	3.954	5.096	6.658
9	1	−3.099	−2.467	−1.997	−1.498	−0.006	1.482	1.980	2.465	3.084
9	2	−3.559	−2.779	−2.222	−1.648	−0.007	1.630	2.212	2.773	3.553
9	3	−4.079	−3.138	−2.491	−1.820	−0.007	1.808	2.462	3.137	4.057
9	4	−4.630	−3.513	−2.743	−1.994	−0.008	1.988	2.723	3.512	4.554
9	5	−5.091	−3.880	−3.015	−2.167	−0.008	2.156	2.988	3.859	5.017
9	6	−5.493	−4.196	−3.268	−2.342	−0.009	2.332	3.223	4.165	5.439
9	7	−5.878	−4.481	−3.496	−2.510	−0.009	2.498	3.470	4.456	5.801
9	8	−6.227	−4.769	−3.716	−2.674	−0.010	2.662	3.692	4.743	6.137
9	9	−6.605	−5.058	−3.941	−2.836	−0.010	2.823	3.916	5.031	6.509
10	1	−2.989	−2.401	−1.954	−1.470	−0.006	1.463	1.951	2.394	2.986
10	2	−3.383	−2.692	−2.149	−1.606	−0.007	1.594	2.144	2.680	3.434
10	3	−3.876	−3.000	−2.382	−1.749	−0.007	1.749	2.371	3.021	3.883
10	4	−4.310	−3.325	−2.613	−1.911	−0.007	1.908	2.606	3.358	4.348
10	5	−4.761	−3.655	−2.849	−2.069	−0.008	2.063	2.839	3.663	4.733
10	6	−5.156	−3.943	−3.072	−2.237	−0.008	2.223	3.065	3.948	5.144
10	7	−5.497	−4.222	−3.296	−2.391	−0.009	2.369	3.289	4.231	5.520

Fixed-G, large-n_G Asymptotic Critical Values (Cont'd)

G	M_G	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
10	8	−5.827	−4.472	−3.498	−2.538	−0.010	2.520	3.491	4.494	5.868
10	9	−6.134	−4.730	−3.698	−2.685	−0.010	2.673	3.690	4.747	6.178
10	10	−6.465	−4.986	−3.898	−2.830	−0.011	2.818	3.889	5.004	6.512
11	1	−2.910	−2.350	−1.913	−1.447	−0.006	1.442	1.904	2.333	2.916
11	2	−3.314	−2.612	−2.101	−1.565	−0.007	1.563	2.081	2.581	3.267
11	3	−3.724	−2.899	−2.303	−1.708	−0.007	1.690	2.284	2.868	3.669
11	4	−4.112	−3.198	−2.519	−1.846	−0.007	1.839	2.491	3.193	4.068
11	5	−4.536	−3.470	−2.746	−1.995	−0.008	1.983	2.711	3.469	4.485
11	6	−4.899	−3.750	−2.959	−2.137	−0.008	2.121	2.919	3.741	4.868
11	7	−5.240	−4.017	−3.157	−2.285	−0.009	2.264	3.112	4.000	5.171
11	8	−5.548	−4.260	−3.346	−2.422	−0.009	2.402	3.307	4.259	5.488
11	9	−5.842	−4.495	−3.524	−2.553	−0.010	2.534	3.495	4.469	5.789
11	10	−6.143	−4.713	−3.710	−2.688	−0.010	2.666	3.675	4.708	6.055
11	11	−6.443	−4.943	−3.892	−2.819	−0.011	2.796	3.855	4.937	6.351
12	1	−2.867	−2.311	−1.889	−1.428	−0.006	1.427	1.884	2.304	2.840
12	2	−3.173	−2.541	−2.057	−1.537	−0.007	1.538	2.044	2.526	3.167
12	3	−3.553	−2.806	−2.236	−1.662	−0.007	1.661	2.226	2.800	3.533
12	4	−3.932	−3.066	−2.426	−1.793	−0.007	1.785	2.420	3.057	3.899
12	5	−4.325	−3.328	−2.621	−1.923	−0.007	1.921	2.613	3.321	4.271
12	6	−4.667	−3.609	−2.814	−2.055	−0.008	2.053	2.804	3.586	4.617
12	7	−5.003	−3.854	−3.008	−2.188	−0.008	2.175	3.000	3.826	4.935
12	8	−5.300	−4.096	−3.188	−2.321	−0.009	2.309	3.174	4.037	5.230
12	9	−5.587	−4.289	−3.365	−2.446	−0.010	2.426	3.349	4.266	5.512
12	10	−5.862	−4.509	−3.541	−2.563	−0.010	2.548	3.508	4.467	5.819
12	11	−6.129	−4.720	−3.703	−2.684	−0.010	2.664	3.682	4.676	6.097
12	12	−6.402	−4.930	−3.868	−2.803	−0.011	2.783	3.846	4.884	6.368
13	1	−2.795	−2.281	−1.869	−1.424	−0.006	1.415	1.864	2.273	2.811
13	2	−3.100	−2.499	−2.018	−1.521	−0.007	1.516	2.016	2.496	3.095
13	3	−3.444	−2.726	−2.191	−1.630	−0.007	1.624	2.184	2.724	3.416
13	4	−3.797	−2.975	−2.370	−1.751	−0.007	1.743	2.354	2.971	3.801
13	5	−4.168	−3.216	−2.556	−1.874	−0.007	1.860	2.532	3.217	4.132
13	6	−4.481	−3.460	−2.724	−1.992	−0.008	1.984	2.721	3.448	4.446
13	7	−4.790	−3.686	−2.913	−2.117	−0.008	2.100	2.891	3.684	4.775
13	8	−5.094	−3.915	−3.090	−2.236	−0.009	2.214	3.064	3.894	5.053
13	9	−5.350	−4.128	−3.248	−2.355	−0.009	2.332	3.230	4.123	5.300
13	10	−5.612	−4.328	−3.408	−2.473	−0.009	2.448	3.393	4.315	5.586
13	11	−5.859	−4.528	−3.559	−2.584	−0.010	2.563	3.540	4.511	5.828
13	12	−6.107	−4.706	−3.717	−2.694	−0.010	2.675	3.690	4.705	6.084
13	13	−6.356	−4.898	−3.869	−2.804	−0.011	2.784	3.841	4.897	6.333
14	1	−2.765	−2.265	−1.846	−1.413	−0.006	1.405	1.843	2.250	2.748
14	2	−3.030	−2.447	−1.995	−1.500	−0.006	1.495	1.979	2.437	3.046
14	3	−3.345	−2.657	−2.151	−1.605	−0.007	1.591	2.140	2.653	3.358
14	4	−3.683	−2.893	−2.315	−1.715	−0.007	1.701	2.289	2.886	3.665
14	5	−3.994	−3.121	−2.477	−1.827	−0.007	1.814	2.451	3.114	4.003
14	6	−4.314	−3.341	−2.640	−1.934	−0.007	1.922	2.624	3.344	4.298
14	7	−4.609	−3.574	−2.817	−2.048	−0.008	2.031	2.787	3.551	4.587
14	8	−4.906	−3.783	−2.982	−2.158	−0.008	2.146	2.959	3.769	4.844
14	9	−5.176	−3.972	−3.128	−2.275	−0.009	2.262	3.103	3.970	5.096
14	10	−5.383	−4.154	−3.278	−2.387	−0.009	2.365	3.249	4.173	5.358
14	11	−5.611	−4.340	−3.423	−2.488	−0.009	2.473	3.402	4.350	5.603
14	12	−5.865	−4.522	−3.566	−2.587	−0.010	2.573	3.546	4.528	5.857
14	13	−6.095	−4.707	−3.715	−2.694	−0.010	2.677	3.683	4.703	6.091
14	14	−6.325	−4.885	−3.856	−2.795	−0.010	2.778	3.822	4.881	6.321

Fixed-G, large-n_G Asymptotic Critical Values (Cont'd)

G	M_G	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
15	1	−2.717	−2.230	−1.834	−1.401	−0.006	1.393	1.834	2.235	2.723
15	2	−2.965	−2.420	−1.965	−1.488	−0.006	1.475	1.952	2.410	2.971
15	3	−3.255	−2.613	−2.106	−1.577	−0.007	1.574	2.089	2.601	3.249
15	4	−3.558	−2.822	−2.257	−1.679	−0.007	1.675	2.241	2.811	3.564
15	5	−3.875	−3.021	−2.417	−1.786	−0.007	1.777	2.394	3.029	3.847
15	6	−4.177	−3.245	−2.570	−1.889	−0.007	1.881	2.551	3.237	4.155
15	7	−4.430	−3.445	−2.733	−1.991	−0.008	1.990	2.711	3.433	4.424
15	8	−4.727	−3.656	−2.887	−2.097	−0.008	2.086	2.863	3.638	4.681
15	9	−4.980	−3.851	−3.023	−2.208	−0.009	2.192	3.010	3.834	4.949
15	10	−5.205	−4.010	−3.171	−2.315	−0.009	2.299	3.159	4.004	5.176
15	11	−5.416	−4.196	−3.312	−2.412	−0.009	2.397	3.293	4.167	5.422
15	12	−5.618	−4.372	−3.443	−2.511	−0.010	2.498	3.428	4.339	5.619
15	13	−5.826	−4.531	−3.578	−2.606	−0.010	2.596	3.555	4.510	5.844
15	14	−6.036	−4.702	−3.704	−2.704	−0.010	2.691	3.688	4.665	6.059
15	15	−6.248	−4.867	−3.834	−2.799	−0.011	2.785	3.817	4.829	6.271
20	1	−2.606	−2.160	−1.780	−1.369	−0.006	1.360	1.781	2.156	2.604
20	2	−2.786	−2.290	−1.875	−1.433	−0.006	1.426	1.871	2.289	2.780
20	3	−2.998	−2.443	−1.986	−1.499	−0.006	1.492	1.969	2.431	2.990
20	4	−3.227	−2.590	−2.099	−1.572	−0.006	1.565	2.080	2.577	3.221
20	5	−3.446	−2.747	−2.209	−1.642	−0.007	1.640	2.197	2.732	3.440
20	6	−3.686	−2.895	−2.320	−1.722	−0.007	1.718	2.300	2.903	3.642
20	7	−3.900	−3.050	−2.438	−1.802	−0.007	1.795	2.420	3.068	3.880
20	8	−4.138	−3.211	−2.551	−1.881	−0.007	1.875	2.538	3.221	4.101
20	9	−4.343	−3.357	−2.666	−1.959	−0.008	1.950	2.651	3.370	4.327
20	10	−4.530	−3.514	−2.788	−2.036	−0.008	2.025	2.769	3.520	4.520
20	11	−4.724	−3.661	−2.905	−2.118	−0.008	2.101	2.880	3.668	4.723
20	12	−4.943	−3.793	−3.013	−2.197	−0.009	2.177	2.994	3.800	4.890
20	13	−5.093	−3.927	−3.126	−2.273	−0.009	2.254	3.096	3.944	5.061
20	14	−5.305	−4.063	−3.230	−2.347	−0.009	2.329	3.195	4.071	5.256
20	15	−5.441	−4.196	−3.328	−2.422	−0.009	2.405	3.296	4.202	5.403
20	20	−6.235	−4.801	−3.823	−2.779	−0.011	2.762	3.785	4.815	6.221
30	1	−2.490	−2.088	−1.731	−1.343	−0.006	1.332	1.732	2.089	2.501
30	2	−2.615	−2.166	−1.794	−1.381	−0.006	1.373	1.796	2.172	2.618
30	3	−2.745	−2.266	−1.867	−1.422	−0.006	1.416	1.857	2.267	2.749
30	4	−2.892	−2.368	−1.936	−1.469	−0.006	1.463	1.925	2.365	2.895
30	5	−3.035	−2.464	−2.013	−1.516	−0.006	1.511	1.992	2.457	3.041
30	6	−3.192	−2.571	−2.087	−1.563	−0.007	1.560	2.069	2.555	3.193
30	7	−3.355	−2.673	−2.163	−1.614	−0.007	1.610	2.150	2.661	3.349
30	8	−3.491	−2.766	−2.238	−1.669	−0.007	1.657	2.223	2.767	3.474
30	9	−3.651	−2.880	−2.312	−1.718	−0.007	1.711	2.293	2.881	3.624
30	10	−3.795	−2.985	−2.392	−1.768	−0.007	1.763	2.366	2.994	3.770
30	11	−3.945	−3.087	−2.464	−1.822	−0.007	1.815	2.441	3.101	3.918
30	12	−4.083	−3.194	−2.545	−1.875	−0.007	1.866	2.520	3.189	4.075
30	13	−4.228	−3.293	−2.622	−1.923	−0.008	1.917	2.598	3.289	4.224
30	14	−4.357	−3.404	−2.701	−1.974	−0.008	1.973	2.683	3.383	4.358
30	15	−4.482	−3.508	−2.779	−2.026	−0.008	2.022	2.762	3.480	4.488
30	20	−5.104	−3.968	−3.140	−2.292	−0.009	2.273	3.115	3.954	5.070
30	25	−5.617	−4.397	−3.480	−2.534	−0.010	2.519	3.453	4.372	5.634
30	30	−6.138	−4.799	−3.804	−2.771	−0.010	2.752	3.780	4.782	6.170

Fixed-G, large-n_G Asymptotic Critical Values (Cont'd)

G	M_G	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
40	1	−2.462	−2.055	−1.709	−1.330	−0.006	1.320	1.711	2.067	2.450
40	2	−2.546	−2.121	−1.756	−1.359	−0.006	1.347	1.754	2.127	2.525
40	3	−2.630	−2.189	−1.802	−1.390	−0.006	1.382	1.798	2.194	2.635
40	4	−2.741	−2.267	−1.859	−1.421	−0.006	1.418	1.854	2.266	2.728
40	5	−2.854	−2.342	−1.914	−1.453	−0.006	1.448	1.907	2.328	2.836
40	6	−2.951	−2.415	−1.972	−1.490	−0.006	1.485	1.958	2.399	2.946
40	7	−3.059	−2.489	−2.028	−1.525	−0.006	1.521	2.013	2.475	3.058
40	8	−3.184	−2.564	−2.085	−1.564	−0.006	1.557	2.063	2.551	3.173
40	9	−3.291	−2.644	−2.140	−1.601	−0.006	1.596	2.120	2.630	3.283
40	10	−3.400	−2.715	−2.195	−1.639	−0.007	1.631	2.177	2.707	3.402
40	11	−3.520	−2.806	−2.249	−1.678	−0.007	1.667	2.232	2.787	3.505
40	12	−3.635	−2.879	−2.310	−1.716	−0.007	1.708	2.289	2.873	3.619
40	13	−3.728	−2.960	−2.368	−1.755	−0.007	1.747	2.339	2.955	3.716
40	14	−3.848	−3.037	−2.423	−1.794	−0.007	1.785	2.399	3.035	3.828
40	15	−3.970	−3.111	−2.481	−1.832	−0.007	1.824	2.456	3.114	3.929
40	20	−4.486	−3.494	−2.773	−2.029	−0.008	2.018	2.747	3.481	4.447
40	25	−4.953	−3.840	−3.054	−2.222	−0.009	2.210	3.014	3.838	4.913
40	30	−5.373	−4.167	−3.310	−2.413	−0.009	2.394	3.277	4.160	5.340
40	35	−5.753	−4.478	−3.557	−2.590	−0.010	2.575	3.523	4.462	5.754
40	40	−6.167	−4.784	−3.805	−2.768	−0.011	2.749	3.762	4.772	6.150
60	1	−2.410	−2.026	−1.697	−1.316	−0.006	1.309	1.686	2.025	2.416
60	2	−2.473	−2.067	−1.719	−1.334	−0.006	1.323	1.714	2.069	2.450
60	3	−2.538	−2.113	−1.751	−1.355	−0.006	1.344	1.746	2.113	2.513
60	4	−2.596	−2.161	−1.784	−1.377	−0.006	1.368	1.780	2.160	2.580
60	5	−2.663	−2.209	−1.817	−1.399	−0.006	1.390	1.814	2.203	2.650
60	6	−2.734	−2.256	−1.855	−1.421	−0.006	1.413	1.850	2.255	2.722
60	7	−2.806	−2.304	−1.890	−1.441	−0.006	1.435	1.882	2.309	2.792
60	8	−2.876	−2.353	−1.927	−1.465	−0.006	1.458	1.920	2.348	2.857
60	9	−2.943	−2.407	−1.965	−1.488	−0.006	1.483	1.953	2.391	2.928
60	10	−3.017	−2.456	−2.005	−1.513	−0.006	1.507	1.992	2.441	3.011
60	11	−3.094	−2.506	−2.039	−1.536	−0.007	1.532	2.029	2.495	3.083
60	12	−3.171	−2.555	−2.078	−1.560	−0.007	1.555	2.062	2.548	3.166
60	13	−3.248	−2.612	−2.114	−1.585	−0.007	1.581	2.101	2.595	3.242
60	14	−3.333	−2.660	−2.154	−1.611	−0.007	1.605	2.139	2.653	3.314
60	15	−3.403	−2.707	−2.189	−1.638	−0.007	1.630	2.178	2.700	3.386
60	20	−3.770	−2.963	−2.379	−1.765	−0.007	1.759	2.363	2.975	3.740
60	25	−4.155	−3.229	−2.570	−1.897	−0.007	1.884	2.555	3.234	4.105
60	30	−4.481	−3.491	−2.776	−2.026	−0.008	2.015	2.748	3.467	4.447
60	35	−4.791	−3.714	−2.957	−2.155	−0.008	2.144	2.931	3.703	4.767
60	40	−5.080	−3.943	−3.137	−2.288	−0.009	2.270	3.098	3.927	5.037
60	45	−5.357	−4.164	−3.308	−2.412	−0.009	2.391	3.271	4.147	5.324
60	50	−5.613	−4.363	−3.468	−2.531	−0.010	2.517	3.441	4.351	5.590
60	55	−5.883	−4.565	−3.636	−2.653	−0.010	2.634	3.601	4.558	5.857
60	60	−6.136	−4.771	−3.798	−2.772	−0.011	2.749	3.760	4.765	6.118
80	1	−2.386	−2.009	−1.684	−1.313	−0.006	1.299	1.678	2.014	2.397
80	2	−2.437	−2.042	−1.705	−1.325	−0.006	1.314	1.698	2.049	2.428
80	3	−2.472	−2.077	−1.725	−1.338	−0.006	1.327	1.723	2.079	2.467
80	4	−2.520	−2.106	−1.750	−1.354	−0.006	1.342	1.748	2.117	2.515
80	5	−2.567	−2.146	−1.776	−1.372	−0.006	1.359	1.773	2.145	2.560
80	6	−2.621	−2.179	−1.800	−1.390	−0.006	1.378	1.796	2.184	2.613
80	7	−2.670	−2.215	−1.824	−1.406	−0.006	1.396	1.821	2.218	2.670
80	8	−2.731	−2.255	−1.854	−1.421	−0.006	1.412	1.848	2.256	2.729
80	9	−2.785	−2.294	−1.881	−1.437	−0.006	1.430	1.873	2.290	2.777

Fixed-G, large-n_G Asymptotic Critical Values (Cont'd)

G	M_G	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
80	10	−2.836	−2.330	−1.907	−1.454	−0.006	1.448	1.898	2.327	2.831
80	11	−2.893	−2.366	−1.934	−1.471	−0.006	1.465	1.922	2.360	2.878
80	12	−2.939	−2.403	−1.960	−1.487	−0.006	1.482	1.953	2.390	2.935
80	13	−2.991	−2.443	−1.989	−1.507	−0.006	1.502	1.980	2.428	2.998
80	14	−3.040	−2.483	−2.020	−1.524	−0.006	1.518	2.006	2.467	3.055
80	15	−3.105	−2.517	−2.047	−1.542	−0.007	1.536	2.035	2.504	3.110
80	20	−3.394	−2.709	−2.189	−1.638	−0.007	1.628	2.174	2.700	3.391
80	25	−3.674	−2.904	−2.332	−1.733	−0.007	1.729	2.306	2.904	3.658
80	30	−3.943	−3.095	−2.474	−1.830	−0.007	1.822	2.456	3.103	3.939
80	35	−4.204	−3.297	−2.621	−1.927	−0.008	1.922	2.600	3.293	4.178
80	40	−4.485	−3.490	−2.776	−2.025	−0.008	2.016	2.742	3.476	4.443
80	45	−4.714	−3.661	−2.912	−2.125	−0.008	2.110	2.882	3.652	4.681
80	50	−4.929	−3.833	−3.043	−2.222	−0.009	2.206	3.011	3.823	4.901
80	55	−5.140	−3.997	−3.186	−2.317	−0.009	2.299	3.143	3.989	5.130
80	60	−5.342	−4.168	−3.307	−2.409	−0.009	2.390	3.269	4.148	5.323
80	65	−5.544	−4.320	−3.425	−2.500	−0.010	2.485	3.390	4.296	5.523
80	70	−5.753	−4.464	−3.554	−2.591	−0.010	2.571	3.515	4.452	5.732
80	75	−5.940	−4.618	−3.673	−2.679	−0.010	2.658	3.638	4.614	5.954
80	80	−6.131	−4.766	−3.795	−2.768	−0.010	2.748	3.757	4.763	6.148
120	1	−2.362	−1.996	−1.675	−1.304	−0.006	1.293	1.667	2.004	2.367
120	2	−2.392	−2.012	−1.685	−1.312	−0.006	1.301	1.678	2.020	2.398
120	3	−2.426	−2.038	−1.702	−1.321	−0.006	1.311	1.693	2.044	2.419
120	4	−2.457	−2.058	−1.715	−1.331	−0.006	1.319	1.710	2.063	2.449
120	5	−2.491	−2.086	−1.733	−1.343	−0.006	1.332	1.729	2.087	2.475
120	6	−2.518	−2.108	−1.749	−1.354	−0.006	1.343	1.746	2.113	2.502
120	7	−2.554	−2.135	−1.764	−1.365	−0.006	1.354	1.765	2.133	2.536
120	8	−2.587	−2.156	−1.783	−1.378	−0.006	1.367	1.780	2.155	2.572
120	9	−2.622	−2.179	−1.798	−1.389	−0.006	1.379	1.793	2.177	2.606
120	10	−2.653	−2.204	−1.818	−1.399	−0.006	1.390	1.810	2.199	2.643
120	11	−2.693	−2.227	−1.834	−1.411	−0.006	1.401	1.829	2.226	2.680
120	12	−2.730	−2.254	−1.853	−1.420	−0.006	1.411	1.847	2.251	2.713
120	13	−2.767	−2.277	−1.873	−1.429	−0.006	1.424	1.865	2.276	2.752
120	14	−2.797	−2.304	−1.889	−1.441	−0.006	1.435	1.880	2.296	2.790
120	15	−2.836	−2.327	−1.908	−1.452	−0.006	1.447	1.901	2.323	2.827
120	20	−3.011	−2.456	−2.002	−1.511	−0.006	1.507	1.991	2.440	3.012
120	25	−3.199	−2.579	−2.097	−1.571	−0.007	1.567	2.082	2.565	3.202
120	30	−3.395	−2.706	−2.190	−1.638	−0.007	1.628	2.175	2.701	3.378
120	35	−3.585	−2.836	−2.283	−1.701	−0.007	1.694	2.266	2.834	3.559
120	40	−3.764	−2.961	−2.379	−1.765	−0.007	1.758	2.357	2.970	3.735
120	45	−3.951	−3.098	−2.475	−1.831	−0.007	1.823	2.452	3.101	3.911
120	50	−4.142	−3.224	−2.569	−1.895	−0.007	1.887	2.550	3.224	4.096
120	55	−4.307	−3.358	−2.671	−1.961	−0.008	1.953	2.648	3.344	4.266
120	60	−4.471	−3.493	−2.772	−2.026	−0.008	2.016	2.740	3.471	4.427
120	65	−4.617	−3.599	−2.862	−2.092	−0.008	2.079	2.835	3.595	4.597
120	70	−4.790	−3.720	−2.952	−2.158	−0.008	2.142	2.923	3.709	4.743
120	75	−4.934	−3.835	−3.044	−2.222	−0.009	2.208	3.008	3.830	4.899
120	80	−5.090	−3.944	−3.132	−2.286	−0.009	2.268	3.097	3.929	5.043
120	85	−5.224	−4.052	−3.227	−2.349	−0.009	2.327	3.181	4.035	5.193
120	90	−5.353	−4.164	−3.308	−2.409	−0.009	2.390	3.266	4.141	5.328
120	95	−5.480	−4.268	−3.387	−2.470	−0.010	2.452	3.358	4.248	5.441
120	100	−5.599	−4.355	−3.465	−2.531	−0.010	2.512	3.438	4.343	5.575
120	105	−5.741	−4.461	−3.550	−2.593	−0.010	2.571	3.518	4.457	5.722
120	110	−5.871	−4.568	−3.631	−2.651	−0.010	2.629	3.600	4.562	5.866
120	115	−6.006	−4.665	−3.709	−2.709	−0.010	2.687	3.679	4.664	5.996
120	120	−6.131	−4.765	−3.789	−2.769	−0.010	2.743	3.760	4.768	6.127